# PRIME FLOWS IN TOPOLOGICAL DYNAMICS

BY

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ABSTRACT

We present some results in topological dynamics and number theory. The number-theoretical results are estimates of the rates of convergence of sequences

$$\left\{(1/n)\sum_{i=0}^{n-1} \chi_{[0,\beta)}(i\alpha):\ n > 0\right\},$$

where $na$ is irrational, $a$ is taken mod 1, and $0 < \beta < 1$. One of these results is used to construct a homorphism $T$ of a compact metric space $X$ such that the minimal flow $(X,T)$ had no nontrivial homomorphic images, i.e. is a prime flow. We define an infinite family of such flows, and describe other interesting properties of these flows.

## Section 1

Certain familiar concepts in number theory have analogues in topological dynamics. In this paper, we shall be concerned with the notion of a minimal prime flow, i.e., a minimal flow with no factors except the trivial ones. Easy examples of such flows are the minimal flows on a prime number of points. Here, we shall construct an infinite prime flow. This example has a surprisingly easy realization; it is essentially obtained by carefully introducing a delay into an irrational rotation on the circle. Another realization in the bisequence space on $\{0,1\}$ is obtained by doubling the ones in a Sturmian sequence. Thus, by starting with an equicontinuous flow with many obvious factors, we can construct a minimal, strictly ergodic flow on a metric space which has no factors.

In Section 1, we introduce the class of flows which will be the subject of our discussion, and indicate preliminary properties such as weak mixing and mini-

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mality. In Section 2, we prove a technical result which is used in Section 3. In Section 2, we also prove some related results about irregularities of distribution in an arbitrary minimal flow, and point out the implications of these results for some problems in diophantine approximation. In Section 3, we prove that our examples possess a property which we call POD, and in Section 4 we show several properties of POD flows, including their primality. For example, a POD flow is disjoint from every minimal flow except extensions of itself, and is disjoint from every power of itself.

We first define the flows which will concern us. A detailed proof of our assertions can be found in [6, Example 1]. By a flow $(X,T)$ we mean a homomorphism $T$ of a compact Hausdorff space. We let $\boldsymbol{K}$ denote $[0,1)$ with addition modulo one. Let $\alpha\in\boldsymbol{K}$ be irrational and $\beta\neq 0$. Define $f\colon \boldsymbol{K}\to\{0,1\}$ by $f(\gamma)=\chi_{[0,\beta]}(\gamma)$ and for $n\in\boldsymbol{Z}$, define $x_0(n)=f(n\alpha)$. Then $x_0\in S=\Pi'_{-\infty}\{0,1\}$ and if $\sigma\colon S\to S$ is the shift on $S$ $(\sigma x(n)=x(n+1))$, and if $X=\mathcal{O}_\sigma(x_0)^-$ (the orbit closure of $x$), then the flow $(X,\sigma)$ is called a Sturmian flow of type $(\alpha,\beta)$. Define $T_\alpha\colon\boldsymbol{K}\to\boldsymbol{K}$ by $T_\alpha(\gamma)=\gamma+\alpha$. We assume $\beta\notin\boldsymbol{Z}\alpha$.

PROPOSITION 1.1 (cf. [6, Th. 4.1]). *The flows $(X,\sigma)$ and $(\boldsymbol{K},T_\alpha)$ are minimal. There is a homomorphism $\rho\colon(X,\sigma)\to(\boldsymbol{K},T_\alpha)$, $\rho(\sigma^n x_0)=n\alpha$, such that $\rho^{-1}(\gamma)$ is a singleton unless $\gamma\in E=(\boldsymbol{Z}\alpha)\cup(\beta+\boldsymbol{Z}\alpha)$, in which case $\rho^{-1}(\gamma)$ is two points. Moreover, if $\rho^{-1}(0)=\{x_0,\bar{x}_0\}$ and $\rho^{-1}(\beta)=\{y_0,\bar{y}_0\}$ with $x_0(0)=y_0(0)=1$, then $\bar{x}_0(0)=\bar{y}_0(0)=0$ and for $n\neq 0$, $x_0(n)=\bar{x}_0(n)$ and $y_0(n)=\bar{y}_0(n)$.*

Next we define the flow induced from $(X,\sigma)$ which will turn out to be a prime flow. Let $B=\{x\in X\mid x(0)=1\}$, and let $A$ be a homeomorphic copy of $B$ such that $X\cap A=\varnothing$, with $\phi\colon B\to A$ a homeomorphism. Let $Y=X\cup A$ and define $\Psi\colon Y\to Y$ by

$$\Psi(y)=\begin{cases}\phi(y) & \text{if } y\in B\\ \sigma(\phi^{-1}(y)) & \text{if } y\in A\\ \sigma(y) & \text{if } y\in X-B.\end{cases}$$

If we ignore most of the "doubled" points in $X$ by assuming that $B$ looks like $[0,\beta]$ and $X-B$ looks like $[\beta,1]$, then a picture of $Y$ might be as shown in Fig. 1, where the action of $\sigma$ on $X$ is given by solid arrows and the action of $\Psi$ by dashed arrows. Note that $y=\sigma^2x=\Psi^4x$.

PROPOSITION 1.2 (cf. [6, remark following Th. 4.1]). *$(Y,\Psi)$ is weakly mixing and minimal.*

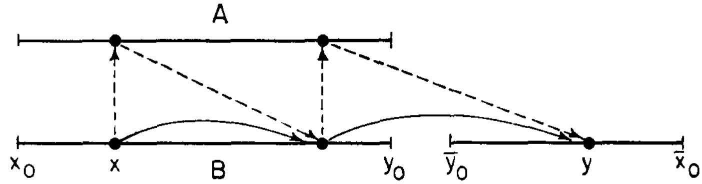


Fig. 1

PROOF. Minimality is clear because $(X,\sigma)$ is minimal.

For weak mixing, let $g$ be a continuous eigenfunction of $(Y,\Psi)$ with eigenvalue $\lambda$, i.e. $g(\Psi(y)) = \lambda\, g(y)$ for $y \in Y$. If $h$ denotes the restriction of $g$ to $X \subseteq Y$, and if we define $u(x) = 1 + x(0)$ for $x \in X$, then $h(\sigma(y)) = \lambda^{u(x)}\, h(y)$. Now it follows from Proposition 1.1 that $x_0, \bar{x}_0$ are positively and negatively proximal, i.e. there are sequences $\{n_k\} \subseteq \mathbf{Z}^+$, $\{m_k\} \subseteq \mathbf{Z}^-$ such that $\lim_k \sigma^{n_k}(x_0) = \lim_k \sigma^{n_k}(\bar{x}_0)$ and $\lim_k \sigma^{m_k}(x_0) = \lim_k \sigma^{m_k}(\bar{x}_0)$. Notice that $u(\sigma^n(x_0)) = u(\sigma^n(\bar{x}_0))$ for $n \neq 0$, so for $n > 0$ we have

$$\frac{h(\sigma^n(x_0))}{h(x_0)} = \frac{h(\sigma^n(x_0))}{h(\sigma^{n-1}(x_0))} \cdots \frac{h(\sigma(x_0))}{h(x_0)}$$

$$= \frac{h(\sigma^n(\bar{x}_0))}{h(\sigma^{n-1}(\bar{x}_0))} \cdots \frac{h(\sigma(\bar{x}_0))}{h(\bar{x}_0)}\, \lambda^{u(\bar{x}_0)-u(x_0)} = \frac{h(\sigma^n(\bar{x}_0))}{h(\bar{x}_0)}\, \lambda^{u(\bar{x}_0)-u(x_0)}.$$

By taking $n = n_k > 0$ and letting $k \to \infty$, we obtain $h(\bar{x}_0) = \lambda^{u(\bar{x}_0)-u(x_0)} h(x_0) = \lambda^{-1} h(x_0)$. A similar calculation with the negative $\{m_k\}$ shows that $h(x_0) = h(\bar{x}_0)$, so we conclude $\lambda = 1$. Since $(Y,\Psi)$ is minimal, this implies $g$ is constant. Thus, every continuous eigenfunction of $(Y,\Psi)$ is constant, so by [4], $(Y,\Psi)$ is weakly mixing.

## Section 2

Our principal goal in this section is a specific result (Corollary 2.3) about the distribution of translates of the sequence $\{n\alpha\} \subseteq \mathbf{K}$. We will obtain this result as a corollary of a more general one concerning an arbitrary minimal flow $(K,T)$ with $K$ compact Hausdorff.

For the next theorem, we fix subsets $A$ and $B$ of $K$, and $x_0 \in K$, where $(K,T)$ is a minimal flow. We are interested in the boundedness of the quantity

$$N(n) = \sum_{i=0}^{n-1} \chi_A(T^i x_0) - \chi_B(T^i x_0).$$

We define $g(T^n x_0) = \chi_A(T^n x_0) - \chi_B(T^n x_0)$ on the orbit of $x_0$, and let $F$ denote the set of points in $K$ to which $g$ *cannot* be extended continuously. Define the bisequence $m_0(n) = g(T^n x_0)$ for integers $n$. Then $m_0$ is an element of the space of bisequences on the symbols $\{0, 1, -1\}$, and we let $M$ denote the $\sigma$-orbit closure of $m_0$ in this bisequence space, where $\sigma$ is the shift, $\sigma m(n) = m(n+1)$.

THEOREM 2.1 *With notation as above, we make the following assumptions:*

1) *$(M, \sigma)$ is minimal.*

2) *The map $\pi(\sigma^n m_0) = T^n x_0$ can be extended to a homomorphism $\pi: (M, \sigma) \to (K, T)$.*

3) *There is $x_1 \in K$ whose orbit does not intersect $F$.*

4) *There is $z \in F$ with $\mathcal{O}(z) \cap F = \{z\}$.*

*Then $N(n)$ is unbounded for $n > 0$.*

PROOF. We will assume that $N(n) = \sum_{i=0}^{n} m_0(i)$ is bounded and will obtain a contradiction. By Lemma 2.2 below, there is some $f \in C(M)$ with

(*) $$m(0) = f(\sigma m) - f(m) \text{ for all } m \in M.$$

Now let $U = \{x \in K : f \text{ assumes more than one value on } \pi^{-1}(x)\}$. Since $f$ may be taken to be integer-valued, $U$ is closed. Notice that $F$ is just the set of points $x$ in $K$ such that $\{m(0) : m \in \pi^{-1}(x)\}$ contains more than one point. Thus $x_1 \notin U$ since $\pi^{-1}(x_1)$ is one point by Assumption 3. By Assumption 4, $\{m(0) : m \in \pi^{-1}(z)\}$ is more than one point, and so it follows from (*) that $z$ or $Tz$ is in $U$. We first consider the case $Tz \in U$. It follows from Assumption 4 that if $n > 0$, then $\{m(n) : m \in \pi^{-1}(z)\}$ is a single point, and this and (*) imply that $T^n z \in U$ for $n > 0$. Since $\{T^n z : n > 0\}$ is dense in $K$, this implies that $K = U$, a contradiction. An analogous argument yields a contradiction in case $z \in U$, so we have shown that no such $f$ exists, and the proof is completed.

LEMMA 2.2 (cf. [2, 14.11]). *If $(M, \sigma)$ is an arbitrary minimal flow and $g \in C(M)$, then a necessary and sufficient condition for there to exist $f \in C(M)$ with $f \circ \sigma - f = g$ is that for some $m_0 \in M$, the sums $\sum_{i=0}^{n} g(\sigma^i m_0)$ are bounded for $n > 0$.*

PROOF. Necessity is clear since $\sum_{i=0}^{n} g(\sigma^i m_0) = f(\sigma^{n+1} m_0) - f(m_0)$. Conversely, suppose that $\sum_{i=0}^{n} g(\sigma^i m_0)$ is bounded for $n > 0$ and define homomorphisms $R$ and $T_s$ (for $s \in \mathbf{R}$) of $M \times \mathbf{R}$ by $R(m, t) = (\sigma m, t + g(m))$ and $T_s(m, t) = (m, t + s)$. Since $R^n(m_0, 0) = (\sigma^n m_0, \sum_{i=0}^{n-1} g(\sigma^i m_0))$ for $n > 0$, it follows that the set of limit points of $\{R^n(m_0, 0) \mid n > 0\}$ is compact, and so it contains a minimal set $N$. If

$p: M \times \boldsymbol{R} \to M$ is the projection, then $p(N) = M$ since $M$ is minimal. Now we claim that for $m \in M$, there is a unique $f(m) \in \boldsymbol{R}$ with $(m, f(m)) \in N$. For if $(m, f(m))$, $(m, f(m) + \delta) \in N$, then by minimality we must have $T_\delta N = N$, so $T_{n\delta} N = N$ for $n \in \boldsymbol{Z}$ and since $N$ is compact, $\delta = 0$. Clearly the function $f$ thus defined is continuous and satisfies $f(\sigma m) - f(m) = g(m)$ for $m \in M$.

Corollary 2.3. *Let* $\boldsymbol{K}$ *denote* $[0,1)$ *considered as the compact group of reals* mod 1, *and pick an irrational* $\alpha \in \boldsymbol{K}$ *and* $0 \neq \beta \in \boldsymbol{K}$. *Fix* $\gamma, \gamma' \in \boldsymbol{K}$ *and set* $A = [\gamma, \gamma + \beta)$, $B = [\gamma', \gamma' + \beta)$. *Then*

$$N(n) = \sum_{i=0}^{n} \chi_A(i\alpha) - \chi_B(i\alpha)$$

*is bounded for* $n > 0$ *iff* $\beta \in \boldsymbol{Z}\alpha$ *or* $\gamma - \gamma' \in \boldsymbol{Z}\alpha$.

Proof. Notice that for $a$, $b \in \boldsymbol{K}$, $[a,b) = \{c \in \boldsymbol{K}: a \leqq c < b\}$ if $a \leqq b$ and $[a,b) = [a,1) \cup [0,b)$ if $b < a$. Define $T_\alpha$: $\boldsymbol{K} \to \boldsymbol{K}$ by $T(x) = x + \alpha$. If $\gamma - \gamma' = k\alpha$, then $A = T_\alpha^k B$ so $|N(n)| \leqq |k|$ for all $n$. Similarly, if $\beta = k\alpha$, then since $\chi_{[\gamma,\gamma')} - \chi_{[\gamma+\beta,\gamma'+\beta)} = \chi_A - \chi_B$ and $T^k[\gamma,\gamma') = [\gamma + \beta, \gamma' + \beta)$ we again have $|N(n)| \leqq |k|$.

Now we assume that $\beta \notin \boldsymbol{Z}\alpha$ and $\gamma - \gamma' \notin \boldsymbol{Z}\alpha$ and will prove that $N(n)$ is unbounded. If either $\gamma - \gamma' - \beta \notin \boldsymbol{Z}^\phi \alpha$ or $\gamma' - \gamma - \beta \notin \boldsymbol{Z}^\phi \alpha$, where $\boldsymbol{Z}^\phi = \boldsymbol{Z} - \{0\}$, then we claim that we can apply the theorem, with $(K, T) = (\boldsymbol{K}, T_\alpha)$ and $x_0 = 0$, to conclude that $N(n)$ is unbounded for $n > 0$.

To this end, we will check each assumption of the theorem. Let $m_0$, $F$, $M$, etc. be as in the theorem, and for $x \in \boldsymbol{K}$, set $h(x) = \chi_A(x) - \chi_B(x)$. We remark that in this case, $F = \{\gamma, \gamma + \beta, \gamma', \gamma' + \beta\}$ is the set of discontinuities of $h$, so $F$ contains at most four distinct points. First we verify Assumption 2: we suppose that $\lim_i \sigma^{n_i} m_0 = m_1$ and that $\xi$, $\xi + \delta$ are cluster points of $\{n_i \alpha\}$, and we must show $\delta = 0$. Since $\lim_i h((n_i + n)\alpha) = m_1(n)$ and $\xi + n\alpha$, $\xi + \delta + n\alpha$ are cluster points of $\{(n_i + n)\alpha\}$ for $n \in \boldsymbol{Z}$, it follows from our previous remark that $h(\xi + n\alpha) = h(\xi + \delta + n\alpha)$ whenever both $\xi + n\alpha$ and $\xi + \delta + n\alpha$ are not in $F$. This implies that $h(x) = h(x + \delta)$ for $x \notin F \cup (F + \delta)$, and for our function $h$, this clearly implies $\delta = 0$. For Assumption 1, if $m_1 = \lim_i \sigma^{n_i} m_0$, then choose $\{k_i\} \subseteq \boldsymbol{Z}$ so that the sequence $\{k_i \alpha + \pi(m_1)\}$ decreases to 0. Since $h(x)$ is right continuous and $F$ is finite, $\lim_i \sigma^{k_i} m_1(n) = \lim_i h(\pi(m_1) + n_i\alpha + n\alpha) = m_0(n)$ for $n \in \boldsymbol{Z}$, so $m_0 \in \mathcal{O}_\sigma(m_1)^-$, proving minimality of $(M, \sigma)$. Assumption 3 is clear since $F$ is finite. For Assumption 4, choose $z = \gamma$ if $\gamma - \gamma' - \beta \notin \boldsymbol{Z}^\phi \alpha$ and $z = \gamma'$ if $\gamma' - \gamma - \beta \notin \boldsymbol{Z}^\phi \alpha$.

The remaining case, when $\beta, \gamma - \gamma' \notin \boldsymbol{Z}\alpha$ and $\gamma - \gamma' - \beta$, $\gamma' - \gamma - \beta \in \boldsymbol{Z}^{\phi}\alpha$, requires a bit more work. Notice that assumptions 1–3 of Theorem 2.1 still hold. We assume $\gamma' - \gamma - \beta = k\alpha$, $k > 0$, the case $k < 0$ being similar. We claim that $\pi^{-1}(\gamma + \beta) = \{m_1, m_2\}$ where (**) $m_2(0) - m_1(0) = m_2(k) - m_1(k) = 1$ and $m_1(n) = m_2(n)$ for $n \neq 0$, $k$. This is because $\{m(n) \mid m \in \pi^{-1}(\gamma + \beta)\}$ is one point unless $\gamma + \beta + n\alpha \in F$, i.e. unless $n = 0$ or $n = k$, and for $x \in \{\gamma + \beta, \gamma + \beta + k\alpha\}$, we have $h(x^-) - h(x^+) = 1$, where $h(x^-) = \mathrm{lm}_{y \uparrow x} h(y), h(x^+) = \lim_{y \downarrow x} h(y)$. Now the proof proceeds as in the proof of Theorem 2.1. If $N(n)$ is unbounded, there is $f \in C(M)$ satisfying (*). By Assumption 3, $\boldsymbol{K} \neq U$. It follows from (*) and (**) that $(f(\sigma^{k+1}m_2) - f(\sigma^{k+1}m_1)) - (f(m_2) - f(m_1)) = \sum_{i=0}^{k} m_2(i) - m_1(i) = 2$, so one of $\pi(m_1)$ or $\pi(\sigma^{k+1}m_1)$ is in $U$. If $\pi(m_1) \in U$, then since $\{m(n) \mid m \in \pi^{-1}(\gamma + \beta)\}$ is one point for $n < 0$, $\{\pi(m_1) + n\alpha \mid n < 0\}$ is in $U$. and since this set is dense, we have the contradiction $\boldsymbol{K} = U$. If $\pi(\sigma^{k+1}m_1) \in U$, then similarly $\boldsymbol{K} = U$, so the proof is completed.

Theorem 2.1 is related to the ergodic theorem in that if $\mu$ is an ergodic invariant probability measure for a flow $(K, T)$ and $\mu(A) = \mu(B)$, then the ergodic theorem implies that for almost all $x_0 \in K$ we have $\lim_{n \to \infty} N(n)/n = 0$. One might then ask whether the quantities

$$\sum_{i=0}^{n-1} (\chi_A(T^i x_0) - \mu(A))$$

are bounded. A partial answer is given by the following theorem.

THEOREM 2.4. *Let $(K, T)$ be a minimal flow, and fix $x_0 \in K$ and $A \subseteq K$. Define $m_0(n) = \chi_A(T^n x_0)$, and let $M$ denote the orbit closure of $m_0$ under the shift $\sigma$. If $(M, \sigma)$ is minimal, then*

$$\sum_{i=0}^{n-1} (\chi_A(T^i x_0) - \delta)$$

*is bounded for $n > 0$ only if* $\exp(2\pi i\, \delta)$ *is the eigenvalue of a continuous eigenfunction of the flow $(M, \sigma)$.*

PROOF. If

$$\sum_{i=0}^{n-1} (\chi_A(T^i x_0) - \delta)$$

is bounded for $n > 0$, then since $(M, \sigma)$ is minimal, we can apply Lemma 2.2 and find a continuous function $h: M \to \boldsymbol{R}$ with

$$\chi_A(T^n x_0) - \delta = h(\sigma^{n+1} m_0) - h(\sigma^n m_0)$$

for $n \in \mathbf{Z}$. Thus $\exp(2\pi i(h(\sigma^{n+1} m_0) - h(\sigma^n m_0)) = \exp(-2\pi i \delta)$, and since $\{\sigma^n m_0 : n \in \mathbf{Z}\}$ is dense in $M$ and $h$ is continuous, it follows that $\exp(2\pi i h(\sigma m)) = \exp(-2\pi i \delta)\exp(2\pi i h(m))$ for all $m \in M$. Thus $H(m) = \exp(-2\pi i h(m))$ is a continuous eigenfunction of $(M, \sigma)$ with eigenvalue $\exp(2\pi i \delta)$.

REMARK 2.5. The main result of [3] is a special case of Theorem 2.4 if we take $(K, T) = (\mathbf{K}, T_\alpha)$, $x_0 = 0$, $A = [0, \beta)$, with $\alpha$ irrational and $\beta \neq 0$. In this case it is proved in [3] that $\sum_{i=0}^{n-1}(\chi_A(T^i x_0) - \beta)$ is bounded if and only if $\exp(2\pi i\, \beta)$ is an eigenvalue of $M$; that is, if and only if $\beta \in \mathbf{Z}\alpha$.

## Section 3

In this section we study the particular flow $(Y, \Psi)$ defined in Section 1. Our goal is to show that it is a POD flow.

DEFINITION 3.1. A flow $(Y, \Psi)$ is called proximal orbit dense, or a POD flow if $(Y, \Psi)$ is totally minimal and whenever $x, y \in Y$ with $x \neq y$, then for some $n \neq 0$, $\Psi^n y$ is proximal to $x$.

REMARK. Recall that a minimal flow $(Y, \Psi)$ is totally minimal iff no factor of $(Y, \Psi)$ is a rotation on a finite ($> 1$) number of points.

DEFINITION 3.2. Recalling the notation of Section 1, define $\theta: Y \to X$ by $\theta(x) = x$ if $x \in X$ and $\theta(x) = \Psi^{-1}(x)$ if $x \in A$, and set $\tau = \rho \circ \theta$, so $\tau: Y \to \mathbf{K}$.

LEMMA 3.3. *Let $x, y \in Y$ with $\tau(x) - \tau(y) = k\alpha$, for $k$ some nonnegative integer. Then there is an integer $m$ with $k \leqq m \leqq 2k$ and $\tau(\Psi^m y) = \tau(x)$.*

PROOF. Notice that for any $y \in Y$ we have $\tau(\Psi^2 y) = \tau(y) + \lambda\alpha$ where $\lambda$ is 1 or 2. The lemma easily follows.

PROPOSITION 3.4. *If $x, y \in Y$, $x \neq y$, and $\tau(x) - \tau(y) = k\alpha$, for $k$ some nonnegative integer, then for some $n \neq 0$, $x$ is asymptotic to $\Psi^n y$.*

PROOF. Notice that by asymptotic, we mean positively or negatively asymptotic. By Lemma 3.3, we can find a nonnegative integer $m$ with $\tau(\Psi^m y) = \tau(x)$. The cases when $x \in \{\Psi^{m+1} y, \Psi^m y, \Psi^{m-1} y\}$ can be handled directly, since points which are equal are asymptotic. We therefore assume that $x \notin \{\Psi^{m+1} y, \Psi^m y, \Psi^{m-1} y\}$ and this along with $\tau(\Psi^m y) = \tau(x)$ implies that 0 or $\beta$ is in the orbit of $\tau(x)$. By carefully considering the possibilities, one sees that there is an integer $j$ with $\tau(\Psi^{j+m} y) = \tau(\Psi^j x) \in \{0, \beta\}$. We consider only the case $\tau(\Psi^j x) = \beta$, the other being similar. Then since $\Psi^j x \notin \{\Psi^{j+m+1}y, \Psi^{j+m}y, \Psi^{j+m-1}y\}$, we must have $\{\Psi^{j+m}y, \Psi^j x\}$ equal to either $\{y_0, \bar{y}_0\}$ or $\{\Psi y_0, \bar{y}_0\}$. Now $\bar{y}_0$ is positively asymptotic to $\Psi y_0$ and negatively asymptotic to $y_0$. This means that $\Psi^j x$ is asymptotic to two distinct points in the set $\{\Psi^{j+m+1}y, \Psi^{j+m}, \Psi^{j+m-1}y\}$, and the proposition follows.

Proposition 3.4 implies that points $(x, y) \in Y \times Y$, satisfying $x \neq y$ and $\tau(x) - \tau(y) \in \mathbf{Z}\alpha$, also satisfy the POD condition. We now turn to the other points in $Y \times Y$. We fix $x_1, y_1 \in Y$ with $\tau(x_1) - \tau(y_1) \notin \alpha\mathbf{Z}$.

LEMMA 3.5. *Let $x \in Y$, and for $n \in \mathbf{Z}^+$ define $c_n(x) = \sum_{k=1}^{n} \chi_X(\Psi^k x)$. Then for $n \in \mathbf{Z}^+$ we have*

$$\sum_{j=0}^{c_n(x)} \theta x(j) = \chi_A(x) + \sum_{j=0}^{n} \chi_B(\Psi^j x).$$

PROOF. We first consider the case $x \in X$. A straightforward induction shows that for $n \geqq 1$, $\sigma^{c_n(x)}x = \Psi^n x$ whenever $\Psi^n x \in X$. This provides the key step in the proof by induction that

$$\sum_{j=0}^{c_n(x)} \chi_B(\sigma^j x) = \sum_{j=0}^{n} \chi_B(\Psi^j x) \text{ for } n \in \mathbf{Z}^+.$$

Noticing that for $x \in X$, $\theta x(j) = \chi_B(\sigma^j x)$, we have proven the lemma if $x \in X$.

Now if $x \notin X$, then $x \in A$, and thus $\Psi^{-1}x \in X$. In this case $c_n(\Psi^{-1}x) = c_{n-1}(x)$, so applying the above result to $\Psi^{-1}x$ we have

$$\sum_{j=0}^{c_{n-1}(x)} \chi_B(\sigma^j \Psi^{-1}x) = \sum_{j=0}^{n} \chi_B(\Psi^{j-1}x) \text{ for } n > 1.$$

Since in this case $\theta(x) = \theta(\Psi^{-1}x)$, it follows that $\chi_B(\sigma^j \Psi^{-1}x) = \theta x(j)$, and furthermore $\chi_B(\Psi^{-1}x) = 1$, so we have shown

$$\sum_{j=0}^{c_{n-1}(x)} \theta x(j) = 1 + \sum_{j=0}^{n-1} \chi_B(\Psi^j x) \text{ for all } n > 1,$$

which proves the lemma in the case $x \in A$, so the proof is completed.

LEMMA 3.6. *The expression $c_n(x_1) - c_n(y_1)$ is unbounded as a function of $n \in \mathbf{Z}^+$.*

PROOF. Suppose on the contrary that $\left| \sum_{j=0}^{n-1} \chi_X(\Psi^j x_1) - \chi_X(\Psi^j y_1) \right| \leqq D$ for $n \in \mathbf{Z}^+$. Then since $\Psi(B) = A$, and $A = Y - X$, we have

$$\left|\sum_{j=0}^{n-1} \chi_B(\Psi^j x_1) - \chi_B(\Psi^j y_1)\right| = \left|\sum_{j=1}^{n} \chi_A(\Psi^j x_1) - \chi_A(\Psi^j y_1)\right|$$

$$= \left|\sum_{j=1}^{n} \chi_X(\Psi^j x_1) - \chi_X(\Psi^j y_1)\right| \leqq D.$$

Now choose $m > 1$; then there is some $n > 1$ with $\sum_{j=1}^{n} \chi_X(\Psi^j x_1) = m$ and we set $\sum_{j=1}^{n} \chi_X(\Psi^j y_1) = m'$, so $|m - m'| \leqq D$. Now pick $n'$ so that $\sum_{j=1}^{n'} \chi_X(\Psi^j y_1) = m$. Since whenever $\Psi^j x \notin X$, then both $\Psi^{j+1}x$ and $\Psi^{j-1}x$ are in $X$, we have $|n - n'| < 2D$. Now taking $x = x_1$ in Lemma 3.5, we have

$$\sum_{j=0}^{m} \theta x_1(j) = \sum_{j=0}^{c_n(x_1)} \theta x_1(j) = \chi_A(x_1) + \sum_{j=0}^{n} \chi_B(\Psi^j x_1),$$

and similarly

$$\sum_{j=0}^{m} \theta y_1(j) = \chi_A(y_1) + \sum_{j=0}^{n'} \chi_B(\Psi^j y_1).$$

Thus

$$\left|\sum_{j=0}^{m} \theta x_1(j) - \theta y_1(j)\right| \leqq 2 + \left|\sum_{j=0}^{n} \chi_B(\Psi^j x_1) - \chi_B(\Psi^j y_1)\right|$$

$$+ \left|\sum_{j=n+1}^{n'} \chi_B(\Psi^j x_1) - \chi_B(\Psi^j y_1)\right| \leqq 2 + 3D.$$

Since $m$ was arbitrary, this contradicts Corollary 2.3 (notice that $\theta x(j) = \chi_{[0,\beta]}(\tau x + n\alpha)$ if $\tau x + n\alpha \notin \{0,\beta\}$). Q.E.D.

LEMMA 3.7. *For any $\omega \in \boldsymbol{K}$ there exist $x_2, y_2 \in \mathcal{O}(x_1, y_1)^-$ with $\tau(x_2) - \tau(y_2) = \omega$.*

PROOF. Notice that $\tau(\Psi^n x) = \tau(x) + c_n(x)\alpha$ for $n \geqq 1$, so $\tau(\Psi^n x_1) - \tau(\Psi^n y_1) = \tau(x_1) - \tau(y_1) + (c_n(x_1) - c_n(y_1))\alpha$. Since $c_n(x_1) - c_n(y_1)$ is unbounded and changes in increments of $\pm 1$, its range contains either $\boldsymbol{Z}^+\alpha$ or $\boldsymbol{Z}^-\alpha$, both of which are dense in $\boldsymbol{K}$, proving the lemma.

PROPOSITION 3.8. *If $\tau(x_1) - \tau(y_1) \notin \boldsymbol{Z}\alpha$, then for some $n \neq 0$, $\Psi^n y_1$ is proximal to $x_1$.*

PROOF. By Lemma 3.7 we can find $(x_2, y_2) \in \mathcal{O}(x_1, y_1)^-$ with $\tau(x_2) - \tau(y_2) \in \boldsymbol{Z}^+\alpha$. Applying Proposition 3.4, we see that for some $n \neq 0$, $x_2$ is proximal to $\Psi^n y_2$. This implies that $x_1$ is proximal to $\Psi^n y_1$, and the proposition is proved.

Together, Propositions 3.4 and 3.8 imply that if $x, y \in Y$ with $x \neq y$, then for some $n \neq 0$, $\Psi^n y$ is proximal to $x$. Furthermore, $(Y, \Psi)$ is totally minimal, since it is weakly mixing and minimal [4, 3.1], and thus we have shown that $(Y, \Psi)$ is POD.

## Section 4

In the previous sections, we have shown the existence of a class of POD flows. In this section, we turn to the general study of such flows. For this section $(Y, \Psi)$, $(X, \phi)$, etc. denote arbitrary minimal flows.

REMARKS.

1. We first note that in a POD flow $(Y, \Psi)$, if $x \in Y$ and $P(x)$ is the set of points proximal to $x$, the $\mathcal{O}(P(x)) = Y$.

2. We will often use the following condition, which is easily seen to be equivalent to POD: $(Y, \Psi)$ is totally minimal and if $(x, y) \in Y \times Y$ with $x \neq y$, then $\mathcal{O}(x, y)^- \supseteq \Gamma_n$ for some $n \neq 0$, where $\Gamma_n = \{(x, \Psi^n x) : x \in Y\}$ is the graph of $\Psi^n$. The equivalence follows since $x$ is proximal to $\phi^n y$ iff $\mathcal{O}(x, \phi^n y)^- \supseteq \{(z, z) \mid z \in Y\}$.

3. A weaker condition than POD is: $(Y, \Psi)$ is totally minimal and the only minimal sets in $Y \times Y$ are the graphs $\Gamma_n$, $n \in \mathbf{Z}$. We do not know whether this weaker condition implies that $(Y, \Psi)$ is prime.

DEFINITION 4.1. A minimal flow $(Y, \Psi)$ is prime if given any homomorphism $\pi: (Y, \Psi) \to (X, \phi)$ onto a nontrivial flow $(X, \phi)$, then $\pi$ is an isomorphism.

Prime flows were first investigated in [5].

THEOREM 4.2. *Let $(Y, \Psi)$ be a POD flow. Then $(Y, \Psi)$ is prime.*

PROOF. Let $\pi$ be as in Definiiion 4.1 and for the purpose of contradiction, assume $\pi$ is not an isomorphism. Define $R = \{(x, y) : \pi(x) = \pi(y)\}$. Then we can pick $(x, y) \in R$, $x \neq y$, and by Remark 2, $R \supseteq \Gamma_n$ for some $n \neq 0$. Then $(x, \Psi^n x) \in R$ and since $Y$ is minimal, this implies that $(X, \phi)$ is a rotation on a finite number of points. This contradicts the assumption that $(Y, \Psi)$ is totally minimal, and the theorem is proven.

Recall that two minimal flows $(Z, \rho)$, $(Y, \Psi)$ are disjoint iff the product $(Z \times Y, \rho \times \Psi)$ is minimal. If $(Z, \rho)$ is a factor of $(Y, \Psi)$ or, equivalently, $(Y, \Psi)$ is an extension of $(Z, \rho)$, then clearly $(Z, \rho)$ and $(Y, \Psi)$ are not disjoint.

THEOREM 4.3. *Let $(Y, \Psi)$ be a POD flow. If $(Z, \rho)$ is not an extension of $(Y, \Psi)$, then $(Z, \rho)$ and $(Y, \Psi)$ are disjoint.*

PROOF. Let $\Delta$ be a minimal subset of $Y \times Z$. We shall show that $\Delta = Y \times Z$. Since $(Z, \rho)$ is not an extension of $(Y, \Psi)$, there exist $x, y \in Y$ with $x \neq y$ and $z \in Z$ with $(x, z)$, $(y, z) \in \Delta$. Then since $(x, \Psi^n x) \in \mathcal{O}(x, y)^-$ for some $n \neq 0$ by Remark 2, we can find $z' \in Z$ with $(x, z')$, $(\Psi^n x, z') \in \Delta$. Now for every $m \in \mathbf{Z}$, $\Psi^m \times \mathrm{id}_Z$ is an isomorphism from $(Y \times Z, \ \Psi \times \rho)$ onto itself. Since $\Psi^n \times \mathrm{id}_Z(\Delta) \cap \Delta$ has been shown to be nonempty, and $\Delta$ is minimal, $\Psi^n \times \mathrm{id}_Z(\Delta) = \Delta$. Finally, fix $z \in Z$. Then for some $x \in Y$, $(x, z) \in \Delta$. It follows that for every integer $j$, $(\Psi^{jn} x, z) \in \Delta$. Since $(Y, \Psi)$ is totally minimal, $\mathcal{O}_{\Psi^n}(x)^- = Y$, and hence $Y \times \{z\} \subseteq \Delta$. Since $z$ is arbitrary, we have $\Delta = Y \times Z$, as desired.

COROLLARY 4.4. *Let $(Y, \Psi)$ be a POD flow, and $(\mathbf{Z}, \rho)$ a minimal flow. Then*

1. *$(Y, \Psi)$ and $(Z, \rho)$ are disjoint iff $(Y, \Psi)$ and $(Z, \rho)$ have no common factors except the trivial flow.*
2. *If $(Z, \rho)$ is prime then either $(Z, \rho)$ is isomorphic to $(Y, \Psi)$ or they are disjoint.*

THEOREM 4.5. *Let $(Y, \Psi)$ be POD. Then* (1) *$(Y, \Psi)$ is coalescent and its automorphism group is $\{\Psi^n : n \in \mathbf{Z}\}$,* (2) *$(Y, \Psi)$ is regular, and* (3) *$(Y, \Psi)$ has no roots, i.e., if $m \neq +1$ then there does not exist a homomorphism $\phi : Y \to Y$ with $\phi^m = \Psi$.*

PROOF.

1) If $\pi : (Y, \Psi) \to (Y, \Psi)$ is an endomorphism, then $R = \{(x, \pi(x)) : x \in Y\}$ is a minimal subset of $Y \times Y$, so $R = \{(x, \Psi^n x) : x \in Y\}$ for some $n$, and thus $\pi(x) = \Psi^n x$ for $x \in Y$.

2) This follows directly from the definition of POD and [1, Th. 3.5].

3) Suppose $m \neq \pm 1$, and $\phi : Y \to Y$ is a homomorphism with $\phi^m = \Psi$. Then $\phi$ is an automorphism of $(Y, \Psi)$, so $\phi = \Psi^n$ for some $n$. This means that $\Psi = \Psi^{mn}$, contradicting the assumption that $(Y, \Psi)$ is totally minimal.

The flows defined in Sections 1–3 arose in trying to determine whether the conjecture that two minimal abelian flows with no common factor except the trivial flow are disjoint is true. If this conjecture is true, then Theorem 4.3 would be true for any prime flow $(Y, \Psi)$. For suppose $(Y, \Psi)$ and $(Z, \sigma)$ are not disjoint. This would mean that $(Y, \Psi)$ and $(Z, \sigma)$ have a common factor $(W, \rho)$. Since $W$ is not a point, and $(Y, \Psi)$ is prime, we must have that $(W, \rho) \simeq (Y, \Psi)$, and thus $(Z, \sigma)$ is an extension of $(Y, \Psi)$.

We finally will show that if $(Y,\Psi)$ is a POD flow and $p > 1$, then $(Y,\Psi)$ is disjoint from $(Y,\Psi^p)$.

DEFINITION 4.6. If $(W,\delta)$ is a flow and $p$ a positive integer, we form $(W_p,\delta_p)$ as follows:

$$W_p = \bigcup_{i=1}^{p} W \times \{i\}, \ \delta_p(w,i) = \begin{cases} (w, i+1), & i \neq p \\ (\delta w, 1), & i = p. \end{cases}$$

(See Fig. 2)

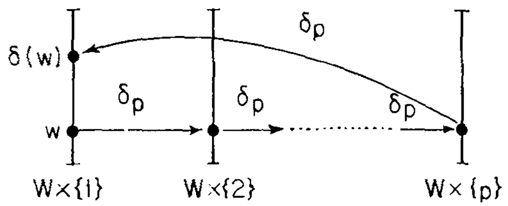


Fig. 2

It is clear that if $(W,\delta)$ is minimal, so is $(W_p,\delta_p)$. Moreover, the minimal flow on $p$ points is a factor of $(W_p,\delta_p)$ via the canonical map $\pi_p\colon W_p \to W$, $\pi_p(w,i) = w$.

LEMMA 4.7. *Let $(Y,\Psi)$ be a POD flow and $p$ an integer with $p > 1$.*

1) *Any factor of $(Y_p,\Psi_p)$ is either a factor of the minimal flow on $p$ points or it admits the minimal flow on $p$ points as a factor.*

2) *No nontrivial factor of $(Y_p,\Psi_p)$ is totally minimal.*

PROOF. Statement 2 of the lemma clearly follows from Statement 1. We denote by $R_p$ the relation corresponding to $\pi_p$, i.e. $R_p = \{\langle (x,i), (y,i)\rangle : x, y \in Y, 1 \leqq i \leqq p\}$. Then we must prove that if $R$ is a closed, $\Psi_p$-invariant equivalence relation on $Y_p$, then either $R \subseteq R_p$ or $R \supseteq R_p$. We will assume $R \nsubseteq R_p$, and will show that $R \supseteq R_p$.

Since $R \nsubseteq R_p$, we can find $x, y \in Y$ and $1 \leqq i < j \leqq p$ with $\langle (x,i),(y,j)\rangle \in R$. Since $(Y,\Psi)$ is POD, we know that for some $n \in \mathbf{Z}$, $(x, \Psi^n x) \in \mathcal{O}_\Psi(x,y)^-$ (if $x = y$, take $n = 0$). Thus $\langle (\Psi^m x, i), (\Psi^m y, j)\rangle = \langle (\Psi_p)^{mp}(x,i), (\Psi_p)^{mp}(y,j)\rangle \in R$ for $m \in \mathbf{Z}$ implies that $\langle (x,i), (\Psi^n x, j)\rangle \in R$. Now $\langle (\Psi^n x, j), (\Psi^{r_1} x, 2j - i \pmod p)\rangle = \langle \Psi_p^{np-i+j}(x,i), \Psi_p^{np-i+j}(\Psi^n x, j)\rangle \in R$, where $r_1 = 2n$ or $2n+1$ depending on whether $2j - i \leqq p$ or $2j - i > p$, respectively, so $r_1$ is defined by $p(r_1 - 2n) < 2j - i \leqq p\,(1 + r_1 - 2n)$. By the transitivity of $R$, $\langle (x,i), (\Psi^{r_1} x, 2j - i \pmod p)\rangle \in R$. Applying this procedure again, $\langle (\Psi^{r_1} x, 2j - i \pmod p), (\Psi^{r_2} x, 3j - 2i \pmod p)\rangle \in R$, where $r_2$ is defined by $p(r_2 - 3n) < 3j - 2i \leqq p\,(1 + r_2 - 3n)$, and again by the transitivity of $R$, we have $\langle (x,i), (\Psi^{r_2} x, 3j - 2i \pmod p)\rangle \in R$. After $k$

steps, we obtain $\langle (x,i), (\Psi^{r_k}x, j + k(j-i) \pmod p)\rangle \in R$, where $p(r_k - (k+1)n) < j + k(j-i) \leqq p(1 + r_k - (k+1)n)$. Notice that $r_k \neq 0$ for sufficiently large $k$. Now choose $k = pq - 1$ where $q$ is large enough so that $r_k \neq 0$. For this choice of $k$, $j + k(j-i) \equiv i (\mathrm{mod}\, p)$, so we have shown that for some $r \neq 0$, $\langle (x,i),(\Psi^r x, i)\rangle \in R$. Now applying $(\Psi_p)^{pr}$ successively to $\langle (x,i), (\Psi^r x, i)\rangle$ and using transitivity of $R$, we obtain after $j$ steps that $\langle (x,i), (\Psi^{jr}x, i)\rangle \in R$. Since $(Y, \Psi^r)$ is minimal (because $(Y,\Psi)$ is POD), it follows that $\langle (x,i), (z,i)\rangle \in R$ for all $z \in Y$. By transitivity of $R$, $\langle (w,i), (z,i)\rangle \in R$ for all $w, z \in Y$. Applying $\Psi_p$ for $p - 1$ successive times, we see that $\langle (w,k), (z,k)\rangle \in R$ for $1 \leqq k \leqq p$, $w, z \in Y$, thus $R \supseteq R_p$. Q.E.D.

THEOREM 4.8. *Let $(Y,\Psi)$ be a POD flow. Then $(Y,\Psi)$ is disjoint from $(Y,\Psi^p)$ for any integer $p > 1$.*

PROOF. First we claim that $(Y,\Psi)$ is disjoint from $(Y_p, \Psi_p)$. If not, then by Theorem 4.3, $(Y,\Psi)$ is a factor of $(Y_p,\Psi_p)$, so by Lemma 4.6, $(Y,\Psi)$ is not totally minimal, a contradiction.

Now let $x, y, w, z \in Y$. Since $(Y \times Y_p, \Psi \times \Psi_p)$ is minimal, for some net $(k_i)$, $\Psi^{k_i}x \to w$ and $(\Psi_p)^{k_i}(y,0) \to (z,0)$. Thus eventually, $k_i = r_i p$ and $(\Psi_n)^{k_i}(y,0) = (\Psi^{r_i}y, 0)$ so $\Psi^{r_i}y \to z$ and $\Psi_i^{pr} \to w$. We have shown that $\mathcal{O}_{\Psi^p \times \Psi}(x,y)$ is dense in $Y \times Y$, so $(Y,\Psi)$ is disjoint from $(Y,\Psi^p)$.

One might wonder just how prevalent POD flows are, within the class of weak mixing minimal flows. If $(X,\phi)$ is the discrete horocycle flow, then $(X,\phi)$ is isomorphic to $(X,\phi^2)$, so by Theorem 4.8, $(X,\phi)$ is not a POD flow.

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